

# Theory of fermion condensation as an analog of the liquid-drop theory of atomic nuclei

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Employing the duality between the momentum distribution  $n(\mathbf{p})$  and density distribution  $\rho(\mathbf{r})$ , problems of theory of systems with flat bands, pinned to the Fermi surface, are discussed. We propose that the Lifshitz topological phase transition associated with the formation of additional pockets of the Fermi surface is the precursor of fermion condensation.

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A theory of strongly correlated Fermi systems, called theory of fermion condensation, was created more than 20 years ago [1–3], (for recent articles, see e.g. [4–11]). A key feature of the phenomenon of fermion condensation is related to *swelling the Fermi surface*: in momentum space, there exists a domain  $\Omega$ , in which the  $T = 0$  single-particle spectrum  $\epsilon(\mathbf{p})$  becomes completely flat:

$$\epsilon(\mathbf{p}) = \mu, \quad \mathbf{p} \in \Omega, \quad (1)$$

where  $\mu$  stands for the chemical potential of the system. The totality of these states is called the fermion condensate (FC), because the  $T = 0$  density of states of Fermi systems with a FC diverges in the same way  $\propto \delta(\epsilon - \mu)$  [5] as the density of states of systems with a Bose condensate.

On the other hand, the spectrum  $\epsilon(\mathbf{p})$  is known to be a variational derivative of the ground-state energy functional  $E(n)$  with respect to the momentum distribution  $n(\mathbf{p})$ . As a result, in three-dimensional homogeneous Fermi liquid (FL), the basic Eq. (1) of theory of fermion condensation can be rewritten as a variational condition [1]

$$\frac{\delta E(n)}{\delta n(p)} = \mu, \quad p_{min} < p < p_{max}, \quad (2)$$

determining the FC momentum distribution  $n_*(p)$  that changes continuously between 0 and 1 inside a momentum interval  $p_{min} < p < p_{max}$ . The FC boundaries  $p_{min}$  and  $p_{max}$  depend on normalization condition. In the strong-coupling limit where the Pauli restriction  $n(p) \leq 1$  is automatically met, it reduces to

$$\rho = \int_0^{p_{max}} n_*(p) \frac{p^2 dp}{\pi^2}. \quad (3)$$

Another key feature of fermion condensation is that its onset is associated with a rearrangement of the topological structure of the ground state. In contrast to conventional FLs, which possess the integer topological charge, beyond the FC transition point its value becomes half-integer, and this topological feature is robust to variations of input parameters [2].

The solutions  $n_*(p)$  of Eq.(2) are found solely in analytically solvable models where the effective energy functional  $E(n)$  has a Hartree-like form

$$E(n) = \sum \epsilon_p^0 n(\mathbf{p}) + \frac{1}{2} \sum \sum U(\mathbf{p} - \mathbf{p}_1) n(\mathbf{p}) n(\mathbf{p}_1), \quad (4)$$

with the interaction  $U(q)$ , possessing a singularity at  $q = |\mathbf{p} - \mathbf{p}_1| \rightarrow 0$ , (see, e.g. [1, 3, 12]). Furthermore, flat bands pinned to the Fermi surface never emerge in iterative numerical calculations of Landau equation [13]

$$\frac{\partial \epsilon(\mathbf{p})}{\partial \mathbf{p}} = \frac{\mathbf{p}}{M} + \int f(\mathbf{p}, \mathbf{p}_1) \frac{\partial n(\mathbf{p}_1)}{\partial \mathbf{p}_1} dv_1, \quad (5)$$

(with  $dv = 2d^3p/(2\pi)^3$ ), except for the models where the interaction functions  $f$  are singular, as mentioned above (see e.g. [5, 14]).

Since numerical calculations of Eq.(5) are practically the sole source of information on the momentum distribution  $n(p)$  beyond the point of a topological rearrangement of the Landau state, concerns were repeatedly voiced that in

Fermi systems with interaction functions regular in momentum space, nontrivial solutions of Eq.(2) are nonexistent, allegedly due to violation of analytic properties, inherent in relevant equations of FL theory. Therefore beyond the critical point, breeding additional Lifshitz pockets of the Fermi surface with FL occupation numbers 0, 1 [14, 15] is the single possible alternative.

However, as we will see, this is not the case. In strongly correlated Fermi systems, phases with the FC do exist irrespective of analytic properties of the input. The analogous situation takes place in the well-studied liquid-drop theory (LDT) of atomic nuclei. Although this theory is formulated in coordinate space, rather than in momentum space, both the spaces are equivalent from the mathematical standpoint, and the momentum distribution  $n(\mathbf{p})$  is dual to the density  $\rho(\mathbf{r})$ . In condensed matter theory, this circumstance is employed already for more than 20 years, (see e.g. [1, 6, 16]), and therefore it is instructive to exploit the duality to clarify the disputed topic.

LDT states are known to emerge as nontrivial solutions of variational equation

$$\frac{\delta E(\rho)}{\delta \rho(\mathbf{r})} = \mu, \quad (6)$$

supplemented with normalization condition

$$N = \int \rho(\mathbf{r}) d\mathbf{r}, \quad (7)$$

where  $N$  stands for a *finite* particle number. Obviously, these two equations are equivalent to Eqs. (2) and (3).

To be more specific, the LDT energy functional  $E(\rho)$  consists of the kinetic energy  $\tau(\rho)$ , proportional to  $\rho^{5/3}(\mathbf{r})$ , and an interaction term, often written in the Skirme-like form

$$E_{int}(\rho) = \frac{1}{2} \int \int V(\mathbf{r}_1, \mathbf{r}_2) \rho(\mathbf{r}_1) \rho(\mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2 + \frac{1}{3} \int \int \int W(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \rho(\mathbf{r}_1) \rho(\mathbf{r}_2) \rho(\mathbf{r}_3) d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3, \quad (8)$$

with interaction potentials  $V$  and  $W$ , vanishing at large distances between interacting particles. For completeness, this expression should have being supplemented with an external field term, say, a gravitational one, however, in the case under consideration, it is of little interest. In spherical systems Eq.(6) reads

$$\mu = \frac{p_F^2(\rho(r))}{2M} + \int V(\mathbf{r}, \mathbf{r}_1) \rho(r_1) d\mathbf{r}_1 + \int \int W(\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2) \rho(r_1) \rho(r_2) d\mathbf{r}_1 d\mathbf{r}_2, \quad (9)$$

with  $p_F(\rho) = (3\pi^2 \rho)^{1/3}$ .

Since  $N$  is finite, the true density  $\rho(r)$  must rapidly fall at  $r \rightarrow \infty$ . This fact is of paramount importance for the analytic structure of solutions of Eq.(9), depending on whether  $\mu$  equals to 0 or not. In the case  $\mu = 0$ , Eq.(9) is met at any distance, implying that the solution  $\rho(r)$  remains finite, analogously to the case of the Rutherford electron atom in the Thomas-Fermi (TF) model [17]. Contrariwise, at  $\mu < 0$  Eq.(9) cannot be satisfied at  $r \rightarrow \infty$ , because the l.h.s. of this equation *remains finite*, while the r.h.s. *comes to nought*. This compels us to conclude that Eq.(9) holds only at  $r < R$ , beyond which  $\rho(r)$  vanishes *identically*. The situation is reminiscent of the situation that takes place in electrically charged atoms, which have a *finite TF radius*  $R$  [17]. Noteworthy, the kinetic term  $p_F^2(\rho)/2M$  possesses the same analytic properties as the density  $\rho$  itself, and therefore its presence on the r.h.s. of Eq.(9) or absence in the limited case  $M \rightarrow \infty$  has no impact on this conclusion.

Explicitly one has

$$\begin{aligned} \mu &= \frac{p_F^2(\rho(r))}{2M} + \int V(\mathbf{r}, \mathbf{r}_1) \rho(r_1) d\mathbf{r}_1 + \int \int W(\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2) \rho(r_1) \rho(r_2) d\mathbf{r}_1 d\mathbf{r}_2, \quad r < R, \\ \rho(r) &\equiv 0, \quad r > R. \end{aligned} \quad (10)$$

To be certain that a nontrivial solution of the variational condition (6) exists, it is sufficient to verify that the above energy functional attains minimum on a simple class of variational functions. If so, nontrivial solutions of Eq.(10) do exist, independent of the presence or absence of singularities in the interaction potentials  $V$  and  $W$ .

The derivative  $\delta E(\rho)/\delta \rho(r)$  is compared with the single-particle energy  $\epsilon(r)$ , while the derivative  $\partial \epsilon(r)/\partial r$  determines the force, acting on a probe particle. Definitely, this force *identically vanishes* in the nuclear interior, while outside the nucleus it reduces to the Coulomb term, produced by the proton component. Bearing in mind the duality, it is instructive to rewrite the spectrum  $\epsilon(r)$  itself in the form

$$\begin{aligned} \epsilon(r) &\equiv \mu, \quad r < R, \\ \epsilon(r) &= \int V(\mathbf{r}, \mathbf{r}_1) \rho(\mathbf{r}_1) d\mathbf{r}_1 + \int \int W(\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2) \rho(\mathbf{r}_1) \rho(\mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2, \quad r > R. \end{aligned} \quad (11)$$

Thereby the function  $\partial\epsilon(r)/\partial r$  is *not analytic*.

Now let us turn to the problem of fermion condensation. Remembering that the quantity  $n(p)$  is dual to the density  $\rho(r)$ , the energy functional of theory of fermion condensation can be written in the form, analogous to that in the LDT,

$$E(n) = \int \frac{p^2}{2M} n(p) dv + \frac{1}{2} \int \int V(\mathbf{p}_1, \mathbf{p}_2) n(p_1) n(p_2) dv_1 dv_2 + \frac{1}{3} \int \int \int W(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) n(p_1) n(p_2) n(p_3) dv_1 dv_2 dv_3, \quad (12)$$

with the single difference: the first kinetic-energy term on the r.h.s. of this equation plays the part of an external harmonic field. The corresponding variational equation (2) has the explicit form

$$\mu = \frac{p^2}{2M} + \int V(\mathbf{p}, \mathbf{p}_1) n_*(p_1) dv_1 + \int \int W(\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2) n_*(p_1) n_*(p_2) dv_1 dv_2, \quad p_{min} < p < p_{max}. \quad (13)$$

supplemented with normalization condition (3). In the strong-coupling limit one has  $p_{min} = 0$ , and therefore  $n(p) = 0$  at  $p > p_{max}$ , otherwise  $n(p) = 1$  at  $p < p_{min}$ , due to the Pauli restriction. Evidently, the quasiparticle momentum distribution  $n(p)$  of the FC problem does not possess properties inherent in analytic functions. Behavior of the group velocity  $\partial\epsilon(p)/\partial p$ , which identically vanishes in the FC region, is analogous to that of the derivative  $\partial\epsilon(r)/\partial r$ , vanishing in the nuclear interior. This implies that the group velocity  $\partial\epsilon(p, n_*)/\partial p$  is not analytic. Definitely, the same is valid for the single-particle spectrum  $\epsilon(p, n_*)$  of the FC problem, given by equation

$$\begin{aligned} \epsilon(p, n_*) &\equiv \mu, \quad p < p_{max}, \\ \epsilon(p, n_*) &= \frac{p^2}{2M} + \int V(\mathbf{p}, \mathbf{p}_1) n_*(p_1) dv_1 + \int \int W(\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2) n_*(p_1) n_*(p_2) dv_1 dv_2, \quad p > p_{max}, \end{aligned} \quad (14)$$

analogous to Eq.(11). Within the context of the Landau approach to theory of Fermi liquid, equation appropriate for numerical calculations of the FC structure has the closed form

$$0 = \frac{p}{M} + \frac{1}{3} \int f_1(p, p_1) \frac{\partial n_*(p_1)}{\partial p_1} \frac{p_1^2 dp_1}{\pi^2}, \quad p < p_{max}, \quad (15)$$

with  $f_1$  standing for the first Legendre harmonic of the Landau interaction function  $f(\mathbf{p}, \mathbf{p}_1)$ , which is a phenomenological input.

It should be emphasized that Eqs.(5) and (15) are *different*. In the strong-coupling limit where the Pauli restriction  $n(p) < 1$  is of minor importance, the domain of the Lifshitz phase diagram, occupied by the phase with the FC, coincides with the domain where Eqs. (2) and/or (15) have nontrivial solution  $n_*(p)$ . In this limit, a different solution  $n_L$ , that satisfies Eq.(5) and consists of a set of additional Lifshitz pockets of the Fermi surface, is of little interest, because the FC distribution  $n_*(p)$  *meets variational condition* (2), while its Lifshitz counterpart  $n_L(p)$ , not. With *decreasing* the coupling constant, the role of the Pauli principle *grows*, so that the Pauli restriction is eventually violated at some critical point where  $n_*(p)$  turns out to be in excess of unity. A plausible scenario for this part of the Lifshitz phase diagram is associated with the formation of a set of Lifshitz pockets. As the coupling constant decreases, the Lifshitz pockets presumably coexist with FC islands until the very last FC island disappears, and the true momentum distribution  $n(p)$  then obeys Eq.(5).

In conclusion, beyond the critical point where the Landau quasiparticle momentum distribution  $n_F(p) = \theta(p_F - p)$  loses its stability, the structure of the true momentum distribution  $n(p)$  evaluated on the base of Eq.(5) is in agreement with a conventional scenario[14, 15, 18–21] for a topological rearrangement of  $n(p)$  that reduces to the generation of a finite number of additional pockets of the Fermi surface. As the coupling constant increases, Eq.(5) and the conventional scenario hold until the onset of fermion condensation, associated with the emergence of nontrivial solutions of Eq.(15) provided these solutions do not violate the Pauli principle. Thereby the Lifshitz topological transition [15] is the *precursor* of fermion condensation, and in the Lifshitz phase diagram of correlated Fermi systems, the phase with additional pockets of the Fermi surface separates the conventional Landau phase from the phase with the fermion condensate.

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